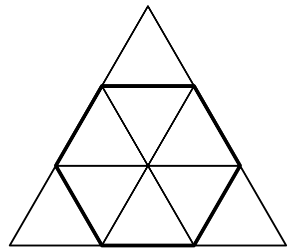
UK Junior Mathematical Olympiad 2009 Solutions

A1 40081 $200^2 + 9^2 = 40000 + 81 = 40081$.

A2 40 cm² Since the hexagon is regular, it has interior angles of 120° and can be dissected into six congruent triangles. The small triangles have three angles of 60° and are therefore equilateral with side equal to that of the hexagon. The three triangles inside the large triangle but outside the hexagon are also equilateral with the same side length as the hexagon. So the area of the hexagon is $\frac{6}{9}\left(= \frac{2}{3}\right)$ of the area of the original triangle.



A3 17 It is clear that each of a, b and c must be less than or equal to 10. A brief inspection will show that the only combination of different square numbers which total 121 is $81 + 36 + 4$.

More formally, the problem can be analysed by considering the remainders after dividing the square numbers less than 121 (1, 4, 9, 16, 25, 36, 49, 64, 81 and 100) by three: the remainders are 1, 1, 0, 1, 1, 0, 1, 1, 0 and 1.

When 121 is divided by 3, the remainder is 1. Therefore $a^2 + b^2 + c^2$ must also leave a remainder of 1. Now we can deduce that two of the three squares must leave a remainder of 0 and so be multiples of 3. There are three square numbers below 121 which are multiples of three: 9, 36 and 81. Checking these, we see that 81 and 36 are the only pair to have a sum which differs from 121 by a perfect square, namely 4. So $a + b + c = 9 + 6 + 2 = 17$.

A4 0 The sum of the first two numbers and the last two numbers is $1004 + 1005 = 2009$. This counts the middle number twice. But the sum of all three numbers is 2009, so the middle number is 0. Hence the product of all three numbers is 0.

[*Alternatively*: Let the three numbers be a, b and c.

We have $$a + b = 1004,$$

$$b + c = 1005$$

and $$a + b + c = 2009.$$

Adding the first two equations gives

$$a + 2b + c = 2009$$

and subtracting the third equation from this gives

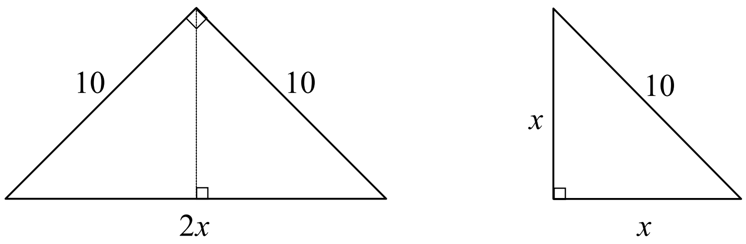
$$b = 0.$$

Thus the product $abc = 0$.]

A5 $18\frac{1}{3}$ The volume of petrol that Andrea put in, as a fraction of the volume of the tank, is the difference between $\frac{2}{3}$ and $\frac{1}{4}$, which is $\frac{5}{12}$. So she put in $\frac{5}{12}$ of 44 litres and $\frac{5}{12} \times 44 = \frac{5 \times 44}{12} = \frac{5 \times 11}{3} = \frac{55}{3} = 18\frac{1}{3}$.

A6 10 cm Since the original triangle is isosceles and right-angled, folding it produces a smaller triangle, also isosceles and right-angled. By Pythagoras' Theorem, the hypotenuse of the original triangle is $\sqrt{200} = 10\sqrt{2}$ cm. Hence the difference between the perimeters of the two triangles is $(10 + 10 + 10\sqrt{2}) - (5\sqrt{2} + 5\sqrt{2} + 10) = 10$ cm.

Alternatively: Let the length of the shorter sides of the new triangle be x cm, shown below. Then the perimeter of the original triangle is $(20 + 2x)$ cm and the perimeter of the new triangle is $(10 + 2x)$ cm. Hence the difference between the perimeters of the two triangles is 10 cm.

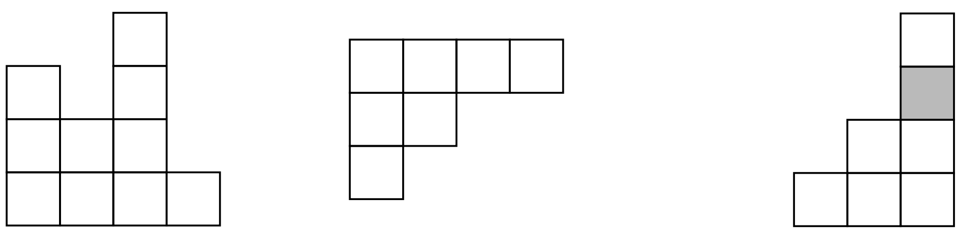


A7 11 As a fraction of 30 minutes, 30 seconds is $\frac{1}{60}$. So we are considering fractions with a denominator of 60. To obtain a fraction of the required form, the numerator must be a factor of 60 (and less than 60). The numerator can therefore be 1, 2, 3, 4, 5, 6, 10, 12, 15, 20 or 30.

A8 22π cm The length of a semicircular arc of radius r is πr and so the total perimeter is $(2 \times (1 + 2 + 4) + 8)\pi = 22\pi$ cm.

A9 133 After every two numbers, one is omitted. Because $89 = 2 \times 44 + 1$, there must be 44 page numbers missing and so the number on the last page is $89 + 44 = 133$.

A10 5000 cm^2 Views from the front and back, the top and bottom, and the two sides are as shown below:



2 lots of 10 faces | 2 lots of 7 faces | 2 lots of 8 faces (one hidden from each side)

Each square face has a surface area of 100 cm^2. Hence the total surface area of Gill's shape is $(20 + 14 + 16) \times 100$ cm^2 $= 5000$ cm^2.

B1 In 2007 Alphonse grew twice the number of grapes that Pierre did. In 2008 Pierre grew twice the number of grapes that Alphonse did. Over the two years Alphonse grew 49 000 grapes, which was 7600 less than Pierre. How many grapes did Alphonse grow in 2007?

Solution

Suppose Pierre grew p grapes in 2007. Then, in 2007, Alphonse grew $2p$ grapes.
Thus, in 2008, Alphonse grew $49\,000 - 2p$ and so Pierre grew $98\,000 - 4p$.
Over the two years, the number of grapes Pierre grew was

$$p + (98\,000 - 4p) = 49\,000 + 7600$$

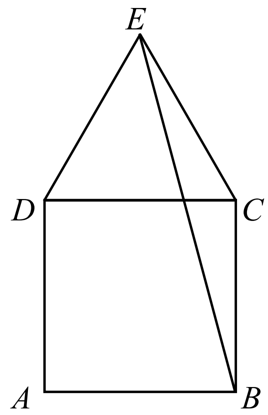
so
$$41\,400 = 3p$$

and
$$p = 13\,800.$$

Hence, in 2007, Alphonse grew $2 \times 13\,800 = 27\,600$ grapes.

B2 *ABCD* is a square. The point *E* is outside the square so that *CDE* is an equilateral triangle. Find angle *BED*.

Solution



Since *ABCD* is a square, $\angle BCD = 90°$; and since *CDE* is an equilateral triangle, $\angle DCE = 60°$.
Thus $\angle BCE = \angle BCD + \angle DCE = 90° + 60° = 150°$.

Because *CDE* is an equilateral triangle, $EC = DC$ and also, because *ABCD* is a square, $DC = CB$. Hence $EC = CB$ and *ECB* is an isosceles triangle.

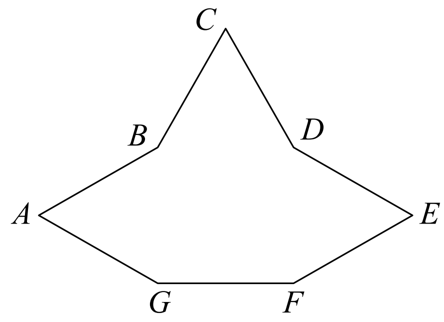
So $\angle CEB = \angle CBE = \frac{1}{2}(180 - 150)° = 15°$, and hence $\angle BED = \angle CED - \angle CEB = 60° - 15° = 45°$.

B3 Tom left a motorway service station and travelled towards Glasgow at a steady speed of 60 mph. Tim left the same service station 10 minutes after Tom and travelled in the same direction at a steady speed, overtaking Tom after a further 1 hour 40 minutes. At what speed did Tim travel?

Solution

Tom travels for 10 minutes longer than Tim, a time of 1 hour and 50 minutes.
Travelling at a speed of 60 mph (or 1 mile per minute), Tom travels a distance of 110 miles.
Tim travelled the same distance in 1 hour and 40 minutes ($1\frac{2}{3}$ hours),
so his speed, in mph, was $110 \div 1\frac{2}{3} = 110 \times \frac{3}{5} = 22 \times 3 = 66$ mph.

B4 The diagram shows a polygon $ABCDEFG$, in which $FG = 6$ and $GA = AB = BC = CD = DE = EF$. Also $BDFG$ is a square. The area of the whole polygon is exactly twice the area of $BDFG$. Find the length of the perimeter of the polygon.

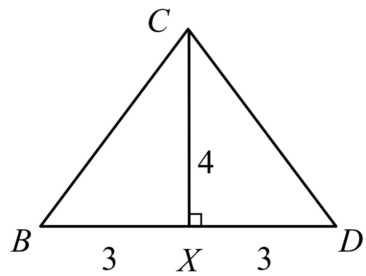


Solution

The area of square $BDFG$ is $6 \times 6 = 36$ square units.
So the total area of the three triangles ABG, BCD and DEF is also 36 square units.
These three triangles are congruent and so each has an area of 12 square units.

The area of each triangle is $\frac{1}{2} \times \text{base} \times \text{height}$ and the base is 6 units and hence we have $\frac{1}{2} \times 6 \times \text{height} = 12$, so the height is 4 units.

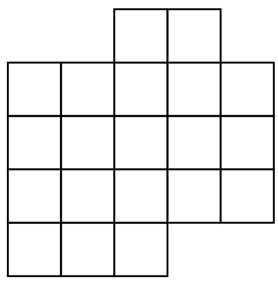
Let X be the midpoint of BD Then CX is perpendicular to the base BD (since BCD is an isosceles triangle).



By Pythagoras' Theorem, $BC = \sqrt{3^2 + 4^2} = 5$ units.

Therefore the perimeter of $ABCDEFG$ is $6 \times 5 + 6 = 36$ units.

B5 An ant wishes to make a circuit of the board shown, visiting each square exactly once and returning to the starting square. At each step the ant moves to an adjacent square across an edge. Two circuits are considered to be the same if the first follows the same path as the second but either starts at a different square or follows the same path in reverse. How many such circuits are possible?



Solution

From Figure 1, we see that there is only one possible route through each of the corner squares, shown shaded, so the ant's path through these squares is as indicated.
There is then only one possible route through square X, giving Figure 2.

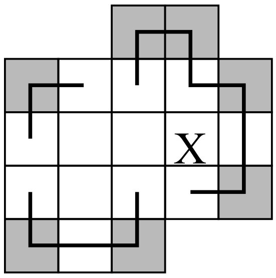


Figure 1

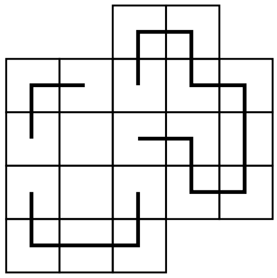


Figure 2

Now consider the shaded square in Figure 3.

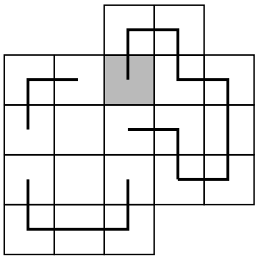


Figure 3

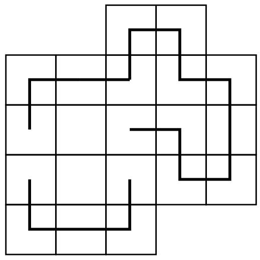


Figure 4

If the path joined this square to the square below, then a closed loop would be formed and the ant could not complete a circuit of the board. Hence the path joins the shaded square to the square on its left (Figure 4).

There are now only two squares which the ant's path has not visited. If a path through all the squares did not join these two, then two loops would be formed instead of a single circuit. We deduce that the path joins these two squares and then there are only two ways of completing the path, as shown in Figure 5.

Figure 5

B6 I want to choose a list of n different numbers from the first 20 positive integers so that no two of my numbers differ by 5. What is the largest value of n for which this is possible? How many different lists are there with this many numbers?

Solution

Any such list contains at most two numbers from the set $\{1, 6, 11, 16\}$, at most two numbers from the set $\{2, 7, 12, 17\}$, and likewise from each of the sets $\{3, 8, 13, 18\}$, $\{4, 9, 14, 19\}$ and $\{5, 10, 15, 20\}$. Hence there are at most $5 \times 2 = 10$ numbers altogether. The list of ten numbers, 1, 2, 3, 4, 5, 11, 12, 13, 14, 15 shows that a selection is indeed possible.

From each of these sets of four numbers of the form $\{a, a + 5, a + 10, a + 15\}$, there are three pairs which do not differ by 5, namely $(a,\ a + 10\}$, $\{a,\ a + 15\}$ and $\{a + 5,\ a + 15\}$. Since we are choosing a pair from each of five such sets, there will be $3^5 = 243$ different lists.